PERGAMON

International Journal of Solids and Structures 36 (1999) 747–761

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

Damage mechanics: basic variables in continuum theories

Alix Cauvin, Rene B. Testa*

Department of Civil Engineering and Engineering Mechanics, Columbia University, New York, NY 10027, U.S.A.

Received 11 June 1997; in revised form 26 January 1998

Abstract

It is shown that any type of material damage that causes stiffness degradation in general requires description by an eighth order damage tensor but that the principle of strain equivalence permits a reduction to a tensor of order four. The actual number of independent damage parameters in such a tensor is related to the material and damage symmetry. For the isotropic case, there must be two independent damage parameters which can be expressed in terms of damage parameter with physical meaning. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

In a broad sense, the term damage refers to degradation or break-up of materials. It can originate from diverse phenomena such as oxidation, carbonation, corrosion, mechanical cleavage, or any type or disintegration or weakening from aging or mechanical processes. In the field of applied mechanics, the pioneering fatigue damage concept proposed by Palmgren (1924), now well-known as the Palmgren–Miner rule, and the creep damage concept of Robinson (1952) have pointed the way to a phenomenological representation of such damage.

It is generally accepted that damage in mechanics can be characterized on the three M-scales: the micro, the meso and the macro. Atomic voids and dislocations are viewed in the microscale, while visible or near-visible discrete damage manifestations, such as loss of section due to corrosion or the isolated cracks treated in fracture mechanics, are seen on the macroscale. The mesoscale is the building block of continuum mechanics in which discrete phenomena can be smeared into mean effects. In the description of damage, the mesoscale serves the same role in that the effects of microvoids, microcracks and other distributed deteriorations are averaged therein. What that

* Corresponding author. Fax: 2128546267.

0020–7683/99/$ - see front matter
PII: S0020–7683(98)00044–4

scale is, specifically depends on inherent dimensions in the material (e.g. reinforcement in a composite and aggregate in concrete) and of the damage itself.

In continuum damage mechanics, the transition from the micro to the meso-scale suffers from the same hurdles that have hindered the step from material science approaches to those of continuum mechanics. On the other hand, the mesoscale formulation will have direct application to the macroscale damage description much as is seen in continuum mechanics and its application to skeletal structures.

Many models have been proposed in recent years to formulate the damage effects on the mesoscale in what has come to be known today as Damage Mechanics. There are two complementary approaches within this body of work: one focuses on the mechanics of the actual damage manifestations, such as microcracking, to determine their combined net effect at the mesoscale. This is the micromechanics approach. The other is phenomenological in that it treats an element with certain properties as if it were in a homogeneous medium without regard to how those properties come about from the damage. It is this latter approach that has been termed continuum damage mechanics and it is the one we follow here, though we shall refer for guidance to the micromechanical concepts as well.

The aim of micromechanical damage theories is to develop models that establish a functional dependence between the random and heterogeneous microstructures and the macro-response of materials. Information concerning distribution, concentration, shapes and orientations of voids and microcracks is needed in these models. The concept of a representative volume element (RVE) Hill (1963), Hashin (1983) at the mesoscale is most important in this approach. Within the RVE the contributions of discrete entities of damage are considered and averages over the RVE give the field of damage variables regarded as internal variables.

The representative volume element (RVE) is significant in the phenomenological approach as well and may be thought of as the typical element of the continuum. Within the RVE, the discrete entities of damage do not appear explicitly, but their effects are represented by means of macroscopic internal variables. One advantage of this simplification of the discrete process of damage is that the derivations can be based on the unifying theory of the thermodynamics of irreversible processes with internal variables, and not solely on physical considerations.

Numerous continuum damage models have emerged in the past twenty years, too many to permit reference to all of them here. The choice of the damage variables is perhaps the most challenging step in the development of these damage models, and many have appeared in the existing literature. In this study, we shall refer to micromechanical arguments to establish the most general form of damage description that unifies all the previous phenomenological models. This formulation establishes, as well, the minimum number of damage parameters required depending on the degree of symmetry of the material properties and of the damage.

2. Description of damage

Within the framework of damage mechanics, only that which causes degradation in the stiffness of a material is considered. Thus, cracking or corrosion in an element can be represented in this approach, but the effects of carbonation in concrete or hydrogen embrittlement in metals may not be included. Nor are plastic deformations unaccompanied by stiffness degradation considered to be damage in this context, although the occurrence of damage can induce additional plastic deformations.

If a material is initially isotropic, and if the effects of damage are reflected in a reduction of the various stiffnesses, they may be designated by reduction factors applied to each stiffness as well as other material properties. In the case where the damaged material remains isotropic these might include

$$\tilde{E} = R_E E, \quad \tilde{G} = R_G G, \quad \tilde{K} = R_K K, \quad \tilde{v} = R_v v \tag{2.1}$$

In these, E, G, K, and v denote, respectively, the Young's modulus, the shear modulus, the bulk modulus, and the Poisson's ratio of the virgin material. If the material becomes anisotropic because of the damage, reduction factors for the various material directions would have to be identified. The scalar factors that apply to stiffnesses must be less than one, i.e. R_E, R_G, $R_K \leqslant 1$ but the effect on other material properties (e.g. R_v) cannot be so categorized, a priori. More commonly, these scalar factors of damage might be described by scalar damage parameters D_i that measure the degree of damage effect in the form:

$$1 - D_i = R_i \tag{2.2}$$

where the subscript i denotes any of the material parameters above. It is these scalar damage measures that are of most interest to us because they are the quantities most readily identified in experiments. But, it is clear that these scalar damage parameters cannot all be independent, just as the material parameters that they modify are not independent of one another but are derived from the elements of the tensor of the elastic constants whose number is limited by the material symmetry. Moreover, these damage parameters may not form the complete set of damage variables because the damage itself may affect the material symmetries and thus cause the number of independent elastic constants to change. Thus, a fundamental set of damage variables should be sought from which the scalars above are derived and which will account for the symmetry or lack of symmetry in the damage, irrespective of the initial material symmetries.

3. Most general damage tensor

At any given state of damage, the elastic portion of the material response will be characterized by a fourth-order tensor $\tilde{\mathbf{E}}$ of the damaged elastic moduli just as the fourth order tensor $\mathbf{E}$ describes the elastic response of the virgin material. In general, one may expect that the damage moduli depend on both the undamaged values and on some measure of the damage level, i.e. $\tilde{\mathbf{E}}$ ($\mathbf{E}$, damage level). Micromechanical theories of composite materials will show the relation of $\tilde{\mathbf{E}}$ to $\mathbf{E}$ as a linear one if the damaged material is considered as a limiting case of such a composite (Cauvin, 1997).

If the relation is shown to be linear, then for two fourth order tensors the linear expression is most generally

$$\tilde{\mathbf{E}} = \mathbf{R_8} :: \mathbf{E} \tag{3.1}$$

The level of damage is contained in the eighth order tensor $\mathbf{R_8}$ and the quadruple inner product is indicated by the operation in eqn (3.1). $\mathbf{R_8}$ contains the level of damage in the sense that it defines what fraction of the stiffnesses remains in the material.

When there is no damage, $\mathbf{R_8}$ must reduce to the identity tensor defined by the relation

$$\mathbf{I_8} :: \mathbf{E} = \mathbf{E} \tag{3.2}$$

Using this unit tensor, it is also possible to rewrite eqn (3.1) so that the actual level of damage is isolated in an eighth-order damage tensor $\mathbf{D_8}$ in the form:

$$\mathbf{R_8} = \mathbf{I_8} - \mathbf{D_8} \tag{3.3}$$

so that

$$\tilde{\mathbf{E}} = (\mathbf{I_8} - \mathbf{D_8}) :: \mathbf{E} \tag{3.4}$$

This eighth-order tensor description of damage with its $(3)^8$ elements, while comprehensive in its generality, would be extremely cumbersome to work with and quite impossible to manage in physical applications. Universal symmetries can reduce the tensor rank and improve tractability.

4. Effective stress and the fourth-order damage tensor

Introduced by Rabotnov (1968) for uniaxial load and extended to the general case by Lemaitre (1971) and Chaboche (1977), the effective stress $\tilde{\boldsymbol{\sigma}}$ is the stress tensor to be applied to a virgin representative volume element in order to obtain the same elastic strain tensor, $\boldsymbol{\varepsilon}^e$, produced by applying the actual stress tensor, $\boldsymbol{\sigma}$, to the damaged volume element. Because the same elastic strain is considered in both damaged and undamaged materials, that strain is considered to be the equivalent strain.

By this definition often called the principle of strain equivalence, the actual stress and effective stress satisfy the equations:

$$\sigma_{ij} = \tilde{E}_{ijkl}\varepsilon^e_{kl} \tag{4.1}$$

$$\tilde{\sigma}_{ij} = E_{ijkl}\varepsilon^e_{kl} \tag{4.2}$$

Using the principle and making use of the symmetry properties inherent in the moduli which make the eighth-order tensor $\mathbf{R_8}$ in eqn (3.1) symmetric in successive pairs of its eight indices, the most general damage description will reduce to a fourth-order damage tensor.

Equation (4.1) with eqn (3.1) and eqn (4.2) give

$$\sigma_{ij} = R_{ijklmnpq}E_{mnpq}E^{-1}_{klrs}\tilde{\sigma}_{rs} = R_{ijrs}\tilde{\sigma}_{rs} \tag{4.3}$$

where the newly introduced fourth-order tensor $\mathbf{R}$ will also possess symmetry in successive pairs of indices.

Starting with eqn (3.3), in which the unit tensor $\mathbf{I_8}$ for the set of tensors with the symmetry possessed by $\mathbf{R_8}$ is (Cauvin, 1997)

$$I_{ijklmnpq} = \tfrac{1}{4}(\delta_{im}\delta_{jn}\delta_{kp}\delta_{lq} + \delta_{im}\delta_{jn}\delta_{kq}\delta_{lp} + \delta_{in}\delta_{jm}\delta_{kp}\delta_{lq} + \delta_{in}\delta_{jm}\delta_{kq}\delta_{lp}) \tag{4.4}$$

it can be shown that $\mathbf{R}$ in eqn (4.3) can be written in the form

$$R_{ijrs} = I_{ijrs} - D_{ijrs} \tag{4.5}$$

with

$$D_{ijrs} = D_{ijklmnpq} E_{mnpq} E^{-1}_{klrs} \tag{4.6}$$

and I_{ijrs} the unit tensor for the set of tensors with the symmetry of **R** is given by

$$I_{ijrs} = \tfrac{1}{2}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) \tag{4.7}$$

Using eqn (4.4) the damaged moduli, eqn (3.4), expressed in terms of the eighth-order damage tensor are given by

$$\tilde{E}_{ijkl} = E_{ijkl} - D_{ijklmnpq} E_{mnpq} \tag{4.8}$$

Postmultiplying eqn (4.6) by **E** and using eqns (4.8) and (4.5) one can obtain

$$\tilde{E}_{ijkl} = E_{ijkl} - D_{ijrs} E_{rskl} = (I_{ijrs} - D_{ijrs}) E_{rskl} = R_{ijrs} E_{rskl} \tag{4.9}$$

The most general description of damage that results in stiffness degradation, therefore, need not be embodied in an eighth-order tensor as in (3.1) or (3.4), but may instead by contained in the fourth-order tensors **R** and **D** of (4.9), as long as the principle of strain equivalence is imposed.

5. Anisotropy of damage

In the virgin state, even in the most general case of anisotropy, there are only 21 independent elements of the fourth-order elastic modulus tensor **E** as a result of the general symmetry requirements

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij} \tag{5.1}$$

where the first three result from the symmetry of the stress and strain tensors and the last from the existence of a strain energy function. With increasing symmetry in elastic properties, the number of stiffness elements decreases until only two remain (the Lamé constants, λ and μ) for the isotropic case. We assume that the material is initially isotropic.

In all likelihood, damage that is caused by an applied stress history will induce anisotropy, but symmetry properties of both the stress history and macroscopic structure of the damaged material can limit the number of independent elements of $\tilde{\mathbf{E}}$ and of the damage tensor **D**. The symmetry of **E** in eqn (5.1) applied to $\tilde{\mathbf{E}}$ as well and dictates a maximum of 21 independent elements. From eqn (4.9), symmetries in the elements of $\tilde{\mathbf{E}}$ imply certain constraint equations on the elements of **D**. Starting with the isotropic **E** and the general $\tilde{\mathbf{E}}$, fifteen constraint equations on the elements of **D** are found from eqn (4.9) to describe general anisotropic damage from an initial isotropic state:

$$(D_{1122} - D_{2211})(1-v) + (D_{1111} - D_{2222})v + (D_{1133} - D_{2233})v = 0 \tag{5.2}$$

$$(D_{1133} - D_{3311})(1-v) + (D_{1111} - D_{3333})v + (D_{1122} - D_{3322})v = 0 \tag{5.3}$$

$$(D_{2233} - D_{3322})(1-v) + (D_{2222} - D_{3333})v + (D_{2211} - D_{3311})v = 0 \tag{5.4}$$

$$D_{1112} = D_{1211}\frac{1-v}{1-2v} + (D_{1222} + D_{1233})\frac{v}{1-2v} \tag{5.5}$$

$$D_{1113} = D_{1311}\frac{1-v}{1-2v} + (D_{1322} + D_{1333})\frac{v}{1-2v} \tag{5.6}$$

$$D_{1123} = D_{2311}\frac{1-v}{1-2v} + (D_{2322} + D_{2333})\frac{v}{1-2v} \tag{5.7}$$

$$D_{2212} = D_{1222}\frac{1-v}{1-2v} + (D_{1211} + D_{1233})\frac{v}{1-2v} \tag{5.8}$$

$$D_{2213} = D_{1322}\frac{1-v}{1-2v} + (D_{1311} + D_{1333})\frac{v}{1-2v} \tag{5.9}$$

$$D_{2223} = D_{2322}\frac{1-v}{1-2v} + (D_{2311} + D_{2333})\frac{v}{1-2v} \tag{5.10}$$

$$D_{3312} = D_{1233}\frac{1-v}{1-2v} + (D_{1211} + D_{1222})\frac{v}{1-2v} \tag{5.11}$$

$$D_{3313} = D_{1333}\frac{1-v}{1-2v} + (D_{1311} + D_{1322})\frac{v}{1-2v} \tag{5.12}$$

$$D_{3323} = D_{2333}\frac{1-v}{1-2v} + (D_{2311} + D_{2322})\frac{v}{1-2v} \tag{5.13}$$

$$D_{1312} = D_{1213} \tag{5.14}$$

$$D_{2312} = D_{1223} \tag{5.15}$$

$$D_{2313} = D_{1323} \tag{5.16}$$

We note that, in general, $D_{ijkl} \neq D_{klij}$ so that the fourth order damage tensor **D** with 21 independent components does not possess the full symmetry of the elastic modulus tensors **E** and $\tilde{\mathbf{E}}$ but it still has the requisite symmetry to place it in the same set of tensors. Additional symmetries in the damaged material will further reduce the number of independent elements of **D**.

5.1. *Orthotropic damage*

If the damaged material is orthotropic at the point in question, then the elasticity tensor will have nine independent elements. The damage that causes this state, starting from the isotropic undamaged material is said to be orthotropic damage and the matrix form of the tensor **D**, from eqn (4.9) and the constraint equations, can be shown to have the form:

$$\mathbf{D} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & 0 & 0 & 0 \\ D_{2211} & D_{2222} & D_{2233} & 0 & 0 & 0 \\ D_{3311} & D_{3322} & D_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2D_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2D_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2D_{1212} \end{bmatrix} \tag{5.17}$$

Although **D** is still not symmetric in this case, the twelve elements appearing in eqn (5.17) are reduced to nine independent elements, just as found for an orthotropic **E** or $\tilde{\mathbf{E}}$, by the three constraint equations (5.2), (5.3), and (5.4). From these, the elements below the main diagonal of the matrix **D** are given by:

$$D_{2211} = D_{1122} + (D_{1111} - D_{2222})\frac{v}{1-v} + (D_{1133} - D_{2233})\frac{v}{1-v} \tag{5.18}$$

$$D_{3311} = D_{1133}\frac{1-v+v^2}{1-v} + (D_{1122} - D_{3333})v + (D_{1111} - vD_{2222} - D_{2233})\frac{v}{1-v} \tag{5.19}$$

$$D_{3322} = D_{2233} + (D_{2222} - D_{3333})v + (D_{1122} - D_{1133})v \tag{5.20}$$

5.2. *Tetragonal damage*

If the orthotropic damage is such that the properties in two directions (say X_2 and X_3) are the same but differ from the properties in the direction X_1, then the damage is said to be tetragonal. For such symmetry in the material properties, the tensor of elastic moduli has six independent elements.

Using eqn (4.9) the elements of the damage tensor are found to be further limited by:

$$D_{1122} = D_{1133} \tag{5.21}$$

$$D_{2233} = D_{3322} \tag{5.22}$$

$$D_{2222} = D_{3333} \tag{5.23}$$

$$D_{1212} = D_{1313} \tag{5.24}$$

In matrix form, then, the tensor **D** for tetragonal damage is

$$\mathbf{D} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1122} & 0 & 0 & 0 \\ D_{2211} & D_{2222} & D_{2233} & 0 & 0 & 0 \\ D_{2211} & D_{2233} & D_{2222} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2D_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2D_{1212} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2D_{1212} \end{bmatrix} \tag{5.25}$$

In this form, D_{2211} is not independent but is given by

$$D_{2211} = D_{3311} = D_{1122}\frac{1}{1-v} + (D_{1111} - D_{2222} - D_{2233})\frac{v}{1-v} \tag{5.26}$$

There are six damage parameters in this case and **D** remains unsymmetric.

5.3. *Hexagonal damage*

The special case in which the elastic properties are the same in all directions in the plane X_2X_3 (isotropic in the plane) is of special interest in applications with axisymmetric geometry, loading and consequent damage. Such symmetry results from hexagonal damage, and both $\tilde{\mathbf{E}}$ and **D** have only five independent elements, which leads to the further reduction of elements of **D** in eqn (5.25) by the relation

$$D_{2323} = \tfrac{1}{2}(D_{2222} - D_{2233}) \tag{5.27}$$

The dependence of D_{2211} on other elements, eqn (5.26), still applies here. The five remaining independent elements of **D** may be denoted simply by the scalars D_1, D_2, D_3, D_4, D_5. In the notation of the (6×6) matrix,

$$\mathbf{D} = \begin{bmatrix} D_1 & D_2 & D_2 & 0 & 0 & 0 \\ D_2' & D_3 & D_4 & 0 & 0 & 0 \\ D_2' & D_4 & D_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & (D_3 - D_4) & 0 & 0 \\ 0 & 0 & 0 & 0 & D_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_5 \end{bmatrix} \tag{5.28}$$

in which eqn (5.26) has given

$$D_2' = \frac{1}{1-v}[D_2 + v(D_1 - D_3) - vD_4] \tag{5.29}$$

Starting from an isotropic undamaged material the hexagonal damaged elastic moduli can be found using eqns (4.9) and (5.28):

$$\tilde{E}_{1111} = (1 - D_1)(\lambda + 2\mu) - 2D_2\lambda \tag{5.30}$$

$$\tilde{E}_{2222} = (1 - D_3)(\lambda + 2\mu) - (D_2' + D_4)\lambda \tag{5.31}$$

$$\tilde{E}_{1122} = (1 - D_1)\lambda - 2D_2(\lambda + \mu) \tag{5.32}$$

$$\tilde{E}_{2233} = (1 - D_3 - D_2')\lambda - D_4(\lambda + 2\mu) \tag{5.33}$$

$$\tilde{E}_{2323} = (1 - D_3 + D_4)\mu \tag{5.34}$$

$$\tilde{E}_{1212} = (1 - D_5)\mu \tag{5.35}$$

in which λ and μ are the Lamé constants of the material in its virgin state. Using the constraint equation given by (5.29), D_2' is eliminated from eqns (5.31) and (5.33) to give

$$\tilde{E}_{2222} = (1-D_3)(\lambda+2\mu) - \frac{\lambda}{1-v}[D_2 + v(D_1 - D_3) + (1-2v)D_4] \tag{5.36}$$

$$\tilde{E}_{2233} = (1-D_3)\lambda - \frac{\lambda}{1-v}\left[D_2 + v(D_1 - D_3) + \frac{1-2v}{v}D_4\right] \tag{5.37}$$

More commonly these material constants are encountered in the (6×6) matrix form for the compliancies

$$\tilde{\mathbf{E}}^{-1} = \begin{bmatrix} \frac{1}{E_1} & -\frac{v_{12}}{E_1} & -\frac{v_{12}}{E_1} & 0 & 0 & 0 \\ -\frac{v_{12}}{E_1} & \frac{1}{E_2} & -\frac{v_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{v_{12}}{E_1} & -\frac{v_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}} \end{bmatrix} \tag{5.38}$$

where the shear compliance in the plane of isotropy X_2X_3 is

$$\frac{1}{2G_{23}} = \frac{1+v_{23}}{E_2} \tag{5.39}$$

Using eqns (5.28), (5.38), (5.39), and (4.9) the damaged moduli in terms of the damage variables are

$$E_1 = E\frac{(1-D_1)(1-D_3-D_4)-2D_2D_2'}{1-D_3-D_4-2vD_2'} \tag{5.40}$$

$$v_{12} = \frac{v(1-D_3-D_4)-(1-v)D_2'}{1-D_3-D_4-2vD_2'} \tag{5.41}$$

$$E_2 = E\frac{(1-D_3+D_4)[(1-D_1)(1-D_3-D_4)-2D_2D_2']}{(1-D_1)(1-D_3-vD_4)-vD_2(1-D_3+D_4)-(1+v)D_2D_2'} \tag{5.42}$$

$$v_{23} = \frac{(1-D_1)(v-vD_3-D_4)+vD_2(1-D_3+D_4)-(1+v)D_2D_2'}{(1-D_1)(1-D_3-vD_4)-vD_2(1-D_3+D_4)-(1+v)D_2D_2'} \tag{5.43}$$

$$G_{23} = \frac{E}{2(1+v)}(1-D_3+D_4) \tag{5.44}$$

$$G_{12} = \frac{E}{2(1+v)}(1-D_5) \tag{5.45}$$

Equation (5.29) must also be satisfied and could be used to eliminate D_2' in these results for hexagonally damaged material.

Finally, the scalar measures of damage of eqn (2.2) might be defined in terms of these material constants so that the damage parameters will have physical meaning, i.e.

$$E_1 = E(1-D_{E_1}) \tag{5.46}$$

$$E_2 = E(1-D_{E_2}) \tag{5.47}$$

$$v_{12} = v(1-D_{v_{12}}) \tag{5.48}$$

$$v_{23} = v(1-D_{v_{23}}) \tag{5.49}$$

$$G_{12} = G(1-D_{G_{12}}) \tag{5.50}$$

one finds the following:

$$D_{E_1} = \frac{D_1(1-D_3-D_4)+2D_2'(D_2-v)}{1-D_3-D_4-2vD_2'} \tag{5.51}$$

$$D_{E_2} = \frac{\alpha_1+\alpha_2}{\beta} \tag{5.52}$$

with

$$\alpha_1(1-D_1)(1-D_3-vD_4)-(1-D_3+D_4)[vD_2+(1-D_1)(1-D_3-D_4)] \tag{5.53}$$

$$\alpha_2 = \frac{D_2}{1-v}[2(1-D_3+D_4)-(1+v)][(D_2+vD_1)-v(D_3+D_4)] \tag{5.54}$$

$$\beta = (1-D_1)(1-D_3-vD_4)-vD_2(1-D_3+D_4)-\frac{1+v}{1-v}D_2[(D_2+vD_1)-v(D_3+D_4)] \tag{5.55}$$

$$D_{v_{12}} = \frac{(1+v)(1-2v)}{v}\frac{D_2'}{1-D_3-D_4-2vD_2'} \tag{5.56}$$

$$D_{v_{23}} = \frac{1+v}{\beta}[D_4[(1-v)(1-D_1)-2vD_2]+D_2(D_2+vD_1-v)] \tag{5.57}$$

$$D_{G_{23}} = D_3 - D_4 \tag{5.58}$$

$$D_{G_{12}} = D_5 \tag{5.59}$$

Again in these, eqn (5.29) can be used to eliminate D_2'. It is in this form that the damage parameters may be evaluated more easily from experiments.

Two more limited forms of the general hexagonal damage theory may be considered.

5.3.1. *Planar transverse isotropy* (*PTI*)

Hexagonal damage corresponding to microcracks randomly distributed in planes parallel to the plane of isotropy X_2X_3 might be the type of damage expected from uniaxial tension in concrete. According to Hoenig (1979), the only moduli affected in this case are E_1 and G_{12}. Therefore, from eqns (5.52), (5.57), and (5.58) we may conclude that the only non vanishing elements of **D** in eqn (5.28) are D_1, D_2' and D_5, and that, moreover, from eqn (5.29)

$$D_2' = \frac{v}{1-v}D_1 \tag{5.60}$$

Damage for this case is then reduced to two independent parameters, D_1 and D_5, and eqn (5.38) reduces to

$$\tilde{\mathbf{E}}^{-1} = \frac{1}{E}\begin{bmatrix} \frac{(1-v)-2v^2D_1}{(1-v)(1-D_1)} & -v & -v & 0 & 0 & 0 \\ -v & 1 & -v & 0 & 0 & 0 \\ -v & -v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1+v) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+v}{1-D_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+v}{1-D_5} \end{bmatrix} \tag{5.61}$$

5.3.2. *Cylindrical transverse isotropy* (*CTI*)

Hexagonal symmetry with cracks randomly distributed in the material but with all the normals to the crack planes orientated in planes parallel to X_2X_3 might be the type of damage expected from uniaxial compression in concrete. According to Hoenig (1979), the only moduli affected in this case are E_2, G_{12}, and G_{23}. Therefore, from eqns (5.30), (5.51), (5.56) we may conclude that the only non vanishing elements of **D** in eqn (5.28) are D_3, D_4, and D_5. Moreover, eqn (5.29) requires that

$$D_3 + D_4 = 0 \tag{5.62}$$

The hexagonal damage theory is thus reduced for CTI to two independent damage variables, D_3 and D_5, and

$$\tilde{\mathbf{E}}^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -v & -v & 0 & 0 & 0 \\ -v & \dfrac{1-(1-v)D_3}{1-2D_3} & -\dfrac{v+(1-v)D_3}{1-2D_3} & 0 & 0 & 0 \\ -v & -\dfrac{v+(1-v)D_3}{1-2D_3} & \dfrac{1-(1-v)D_3}{1-2D_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1+v}{1-2D_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1+v}{1-D_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1+v}{1-D_5} \end{bmatrix} \tag{5.63}$$

5.4. *Cubic damage*

The case of tetragonal damage symmetry with material properties the same in three orthogonal directions reduces the number of elastic parameters to three. One finds

$$D_{1111} = D_{2222} \tag{5.64}$$

$$D_{2233} = D_{2211} = D_{1122} \tag{5.65}$$

and the damage tensor $\mathbf{D}$ which now becomes symmetric also has only three independent elements.

5.5. *Isotropic damage*

For isotropy of the damaged material, $\tilde{\mathbf{E}}$ reduces to only two independent elements. This leads to

$$D_{1212} = \tfrac{1}{2}(D_{1111} - D_{1122}) \tag{5.66}$$

which leaves only two independent parameters, henceforth denoted simply by D_1 and D_2 instead of the more cumbersome D_{1111} and D_{1122}, to describe isotropic damage. In matrix form, the isotropic damage tensor is

$$\mathbf{D} = \begin{bmatrix} D_1 & D_2 & D_3 & 0 & 0 & 0 \\ D_2 & D_1 & D_2 & 0 & 0 & 0 \\ D_2 & D_2 & D_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (D_1-D_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & (D_1-D_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & (D_1-D_2) \end{bmatrix} \tag{5.67}$$

or, in compact form,

$$D_{ijkl} = D_2\delta_{ij}\delta_{kl} + \tfrac{1}{2}(D_1 - D_2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{5.68}$$

Even if the damage at a point in a body leaves the material with the highest level of symmetry, namely isotropy, two independent scalars, D_1 and D_2, eqn (5.68), are needed to characterize that damage. The physical meaning of these parameters is not at all obvious from this form of the damage. They must be written in terms of the degradation of physically meaningful moduli in order to permit interpretation and evaluation.

In a damaged material that remains isotropic, expressions for the damaged moduli can be written in terms of D_1 and D_2 using eqns (4.9) and (5.68)

$$\tilde{\lambda} = \lambda\left(1 - D_1 - \frac{D_2}{v}\right) = (1 - D_\lambda)\lambda \tag{5.69}$$

$$\tilde{\mu} = \mu(1 - D_1 + D_2) = (1 - D_S)\mu \tag{5.70}$$

In each of these, the various combinations of D_1 and D_2 that appear could be identified as scalar damage parameters denoted by D_λ and D_S, but the most useful are those with physical significance such as the shear damage D_S above and D_K, D_E, D_v associated with the bulk modulus, the Young's modulus and Poisson's ratio which can be obtained by using eqns (5.69) and (5.70):

$$\tilde{K} = K(1 - D_1 - 2D_2) = (1 - D_K)K \tag{5.71}$$

$$\tilde{E} = E\frac{(1 - D_1 - 2D_2)(1 - D_1 + D_2)}{(1 - D_1) - (1 + 2v)D_2} = (1 - D_E)E \tag{5.72}$$

$$\tilde{v} = \frac{(1 - D_1)v - D_2}{(1 - D_1) - (1 + 2v)D_2} = (1 - D_v)v \tag{5.73}$$

Of special interest are the damage parameters associated with the shear and the bulk moduli, D_S and D_K, and they are obtained by using eqns (5.70) and (5.71):

$$D_S = D_1 - D_2 \tag{5.74}$$

$$D_K = D_1 + 2D_2 \tag{5.75}$$

6. Conclusion

The physical interpretation of the concept of damage in one-dimension is quite straightforward in terms of area loss and consequent stiffness degradation (Hult, 1987; Rabotnov, 1969). The same cannot be said in extending the concept to three dimensions, and indeed, the selection of the variables to describe internal damage is one of the vexing problems of continuum damage mechanics.

The present work has accomplished several key steps. First, within the same framework as the theory for the analysis of composite materials, it is reasoned that the degraded elastic constants can be expressed with a linear dependence on the undamaged constants, something that occurs automatically in the one dimensional model. Consequently, the damage is expressible in its most general form by a rather intractable eighth-order tensor. Secondly, we have shown that the most general description of damage reduces to a fourth-order tensor if one employs the strain equivalence principle. And finally, we have shown that the elements of the damage tensor are not all independent; moreover, although the symmetries in material properties and physical damage do not appear as obviously in the tensor **D** as they do in the modulus tensor **E** or $\tilde{\mathbf{E}}$, the effects of such symmetries can be expressed in a reduction of the number of independent damage parameters.

We have focused on the case of hexagonal symmetry that may hold particular interest in applications to concrete and on the isotropic damage. Especially noteworthy are the results that two damage parameters are needed to describe isotropic damage. This contrasts with published applications that have extended the single damage parameter description uniaxial reponse to three-dimensional cases. In Cauvin and Testa (1997) we explore the significance of, and the limitations on those two damage parameters.

There remains also the quantitative evaluation of damage parameters from actual tests. This is an endeavour that has been greatly facilitated by the present work by means of the physically meaningful damage parameters that are given in terms of the basic elements of **D**.

References

Aboudi, J., 1991. Mechanics of composite materials. A Unified Micromechanical Approach. Elsevier.

Budiansky, B., O'Connell, R., 1975. Elastic moduli of a cracked solid. Int. J. Solids Structures 12, 81–97.

Carol, I., Rizzi, E., Willam, K., 1994. A unified theory of elastic degradation and damaged based on a loading surface. Int. J. Solids Structures. 31 (20), 2835–2865.

Cauvin, A., 1997. Damage mechanics: basic variables in continuum theories. Doctoral Dissertation. Columbia University.

Cauvin, A., Testa, R.B. 1997. Elastoplastic Material with Isotropic Damage, to be published.

Chaboche, J.L., 1979. Le concept de contrainte effective appliqué à l'elasticité et à la viscoplasticité en présence d'un endommagement anisotrope. In Comportement Mécanique des Solides Anisotropes. EUROMECH Colloque 115. Martinus Nijhoff. pp. 737–760.

Chaboche, J.L., 1992. Damage induced anisotropy: on the difficulties associated with the active/passive unilateral condition. International Journal of Damage Mechanics 1, 148–171.

Chaboche, J.L., 1993. Development of continuum damage mechanics for elastic solids sustaining anisotropic and unilateral damage. International Journal of Damage Mechanics 2, 311–329.

Cordebois, J.P., Sidoroff, F., 1979. Damaged induced elastic anisotropy. In Comportement Mécanique des Solides Anisotropes. EUROMECH Colloque 115. Martinus Nijhoff. pp. 761–774.

Hashin, Z., 1983. Analysis of composite materials. Journal of Applied Mechanics 50, 481–505.

Hill, R., 1963. Elastic properties of reinforced solids: some theoretical principles. J. Mech. Phys. Solids. 11, 357–372.

Hoenig, A., 1979. Elastic moduli of a non-randomly cracked body. Int. J. Solids Structures 15, 137–154.

Hult, J., 1987. Introductions and general overview. In Krajcinovic, D., Lemaitre, J. (Eds.) Continuum Damage Mechanics. Theory and Applications. Springer-Verlag. pp. 1–36.

Kachanov, L.M., 1958. Time of the rupture process under creep conditions. IVZ Akad Nauk, S.S.R., Otd Tech Nauk 8, 28–31.

Krajcinovic, D., Sumarac, D., 1987. Micromechanics of the damage processes. In Krajcinovic, D., Lemaitre J. (Eds.), Continuum Damage Mechanics. Theory and Applications. Springer-Verlag. pp. 135–194.

Krajcinovic, D., Micromechanical basis of phenomenological models. In Krajcinovic, D., Lemautre, J. (Eds.), Continuum Damage Mechanics. Theory and Applications. Springer-Verlag. pp. 195–206.

Lemaitre, J., Chaboche, J.-L, 1990. Mechanics of Solid Materials. Cambridge University Press.

Lemaitre, J., 1992. A Course on Damage Mechanics. Springer-Verlag.

Miner, M. A., 1945. Cumulative damage in fatigue. J. Appl. Mech. 12, A159–A164.

Murakami, S., 1987. Anisotropic aspects of material damage and application of continuum damage mechanics. In Krajcinovic, D., Lemaitre J. (Eds.) Continuum Damage Mechanics. Theory and Applications. pp. 91–133.

Murakami, S. 1988. Mechanical modeling of material damage. ASME J. Appl. Mech. 55, 280–286.

Nemat-Nasser, S., Hori, M., 1990. Elastic solids with microdefects. In Weng, G.T., Taya, M., Abé, H. (Eds.), Micromechanics and Inhomogeneity. The Toshio Mura Anniversary Volume. Springer-Verlag. pp. 297–320.

Palmgren, A., 1924. Die Lebensdauer von Kugellagern. Zeitschrift VDI 68, 339–341.

Rabotnov, Y.N., 1969. Creep rupture. In Hetenyi, M. Vincenti, W.G. (Eds.), Proc. 12 Int. Congr. Appl. Mech. (Stanford, 1968). Springer-Verlag, Berlin, pp. 342–349.

Robinson, E.L., 1952. Effect of temperature variation on the long-time rupture strength of steels. Trans. ASME 74, 777–780.

Sidoroff, F., 1981. Description of anisotropic damage application to elasticity. IUTAM Colloquium, Physical Nonlinearities in Structural Analysis, pp. 237–244.

Simo, J.C., Ju, J.W., 1987a. Strain- and stress-based continuum damage models—I. Formulation. Int. J. Solids Structures 23, 821–840.

Simo, J.C., Ju, J.W. 1978b. Strain- and stress-based continuum damage models—II. Computational aspects. Int. J. Solids Structures 23, 841–869.

Talreja, R., 1994. Damage characterization by internal variables. In Talreja, R. (Ed.), Damage Mechanics of Composite Materials. Elsevier. pp. 53–78.